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Wave motion in a viscous fluid of variable depth

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The linearized equations for wave motion of frequency ω in a shallow, viscous liquid of variable depth h are reduced to a partial differential equation, $\mathscr{L}Z = 0$, for the complex amplitude Z of the free-surface displacement on the assumptions of no slip at the bottom and Kh, $K\delta_* \ll 1$, where $K \equiv \omega^2/g$, and $\delta_* \equiv (\nu/2\omega)^{\frac{1}{2}}$ is a viscous lengthscale. It is shown that capillarity must be included in order to avoid an irregular singular point (which would imply the total absorption of an incoming wave) at $h = 0$. $\mathscr{L}Z$ then is fourth-order and has a regular singular point of exponents $2, 1, 0, 0$ for $h \sim \sigma x \downarrow 0$. The requirements that the free-surface displacement and the shear force be bounded as $h \downarrow 0$ rule out the solutions of exponent 0 and imply a stationary contact line. This last prediction is supported by laboratory observation but is not consistent with the observed runup of long, non-breaking waves on real beaches (for which the condition of no slip presumably must be relaxed). The dissipation for sufficiently small capillarity and viscosity is equal to that calculated from a boundary-layer approximation (despite the violation of the assumption $h \gg \delta_*$ on which that approximation is based). The viscous modification of the Stokes edge wave on a uniform, gentle slope is calculated through matched asymptotic approximations to the solution of $\mathscr{L}Z = 0$.

1. Introduction

I consider here linear wave motion of time dependence $\exp(-\mathrm{i}\omega t)$ in a liquid of depth $h(x,y)$, density ρ, kinematic viscosity ν and surface tension T on the assumptions that

$$Kh \ll 1, \quad K\delta_* \ll 1, \tag{1.1a, b}$$

where

$$1/K \equiv g/\omega^2, \quad \delta \equiv (\nu/\omega)^{\frac{1}{2}} \mathrm{e}^{\frac{1}{4}\mathrm{i}\pi} \equiv (1+\mathrm{i})\,\delta_*, \quad \lambda \equiv (T/\rho g)^{\frac{1}{2}} \tag{1.2a–c}$$

are the relevant lengthscales. The frequency ω has an $O(\delta)$ imaginary part that is significant (in the present approximation) only in the exponent $-\mathrm{i}\omega t$, and the symbol ω is to be regarded as real in the subsequent development except as noted. The domain of primary interest is $h \approx \sigma x = O(\delta)$, and $K\delta_* \ll 1$ must be replaced by $K\delta_* \ll \sigma^2$ in that domain if $\sigma \ll 1$.

I begin, in §2, by constructing a formal operational equation for the complex amplitude of the free-surface displacement on the assumption of no slip at the bottom and $K\delta_* \ll 1$, but without the restriction $Kh \ll 1$. This result may be of some interest as a generalization of the corresponding result for inviscid flow (Miles 1985). In §3, I invoke $Kh \ll 1$ to obtain a generalization, $\mathscr{L}Z = 0$, of the conventional shallow-water equation (Lamb 1932, §193). If $h \gg \delta_*$ this generalization reduces to the replacement of h by $h - \delta$ and is equivalent to a boundary-layer approximation in which the complex parameter δ comprehends both dissipation and viscous dispersion. But if $h \sim \sigma x \downarrow 0$ at a straight shoreline and capillarity is neglected,

$h = 0$ is an irregular singular point of $\mathscr{L}Z$, and Z has an essential singularity, in consequence of which an incoming wave would be totally absorbed at that boundary. Capillarity raises the order of $\mathscr{L}Z$ from two to four, renders the singular point at $h = 0$ regular, and (under the restrictions that Z and the shear force be bounded) implies a fixed contact line. In §5, I calculate the viscous dissipation for $0 < K\lambda \ll 1$ and show that it is equal to that calculated from a boundary-layer approximation despite the violation of $h \gg \delta_*$ near $h = 0$. Finally, I apply the present results to the Stokes edge wave in §6.

The formulation in §§2 and 3 is for a clean surface. In §4, I give the corresponding results for an inextensible (fully contaminated) surface, for which the condition of zero tangential stress is replaced by the condition of zero tangential velocity.

The present model appears to be adequate for laboratory wave scales in the range (periods $\lesssim$ 1 s) of those of Mahony & Pritchard (1980), but it does not provide an adequate description of the shoreline motion on real beaches. In particular, the prediction of a fixed contact line is supported by Mahony & Pritchard's observations but is not consistent with the observed runup of long waves on real beaches, for which both slip and nonlinearity presumably must be accommodated.

2. The boundary-value problem

The continuity and linearized Navier–Stokes equations are (Lamb 1932, §328)

$$\nabla\cdot\boldsymbol{q} = 0, \quad \boldsymbol{q}_t = -\nabla\left(\frac{p}{\rho}+gz\right)+\nu\nabla^2\boldsymbol{q} \quad (-h < z < \zeta), \tag{2.1 a, b}$$

where $\boldsymbol{q} = (\boldsymbol{u}, w)$ is the velocity, $\boldsymbol{u}$ and w are the horizontal and vertical components thereof, p is the pressure, gz is the gravitational potential, and ζ is the free-surface displacement. The linearized free-surface conditions are

$$w = \zeta_t, \quad -p+2\rho\nu w_z = T\nabla^2\zeta, \quad \rho\nu(\boldsymbol{u}_z+\nabla w) = 0 \quad (z = \zeta), \tag{2.2 a–c}$$

corresponding, respectively, to continuity of particle displacement, normal stress, and tangential stress. The bottom (no-slip) conditions are

$$\boldsymbol{u} = w = 0 \quad (z = -h). \tag{2.3}$$

Introducing the complex scalar and vector potentials Φ and $\boldsymbol{A}$ according to

$$\left(\boldsymbol{q}, \frac{p}{\rho}+gz, \zeta\right) = \mathrm{Re}\{(\nabla\Phi+\nabla\times\boldsymbol{A}, \mathrm{i}\omega\Phi, Z)\,\mathrm{e}^{-\mathrm{i}\omega t}\}, \tag{2.4}$$

where Re implies *the real part* of, and invoking (2.1 a, b) and $\nu = -\mathrm{i}\omega\delta^2$, we obtain

$$\nabla^2\Phi = 0, \quad \delta^2\nabla^2\boldsymbol{A} = \boldsymbol{A}. \tag{2.5 a, b}$$

Eliminating the pressure from (2.2 b) with the aid of (2.4), projecting onto $z = 0$, and invoking $T = \rho g\lambda^2$ and $\nu = -\mathrm{i}\omega\delta^2$, we obtain

$$W = -\mathrm{i}\omega Z, \quad \Phi+2\delta^2 W_z = (g/\mathrm{i}\omega)(1-\lambda^2\nabla^2)Z, \quad \delta^2(\boldsymbol{U}_z+\nabla W) = 0 \quad (z = 0) \tag{2.6 a–c}$$

and

$$\boldsymbol{U} = W = 0 \quad (z = -h), \tag{2.7}$$

where $\boldsymbol{U}$ and W are the complex amplitudes of $\boldsymbol{u}$ and w.

Guided by the formulation of the corresponding inviscid problem (Miles 1985), we pose the solutions of (2.5*a*, *b*) in the forms

$$\Phi = \cosh kz\Phi_0(\boldsymbol{x}) + k^{-1}\sinh kz\Phi_1(\boldsymbol{x}) \tag{2.8a}$$

and† $$\boldsymbol{A} = \boldsymbol{z}_1 \times [\cosh \kappa z \boldsymbol{\Psi}_0(\boldsymbol{x}) + \kappa^{-1}\sinh \kappa z \boldsymbol{\Psi}_1(\boldsymbol{x})] \quad (\boldsymbol{z}_1 \cdot \boldsymbol{\Psi}_{0,1} \equiv 0), \tag{2.8b}$$

where $$\boldsymbol{x} \equiv (x, y), \quad k^2 \equiv -(\partial_x^2 + \partial_y^2), \quad \kappa^2 \equiv \delta^{-2} + k^2, \tag{2.9a–c}$$

the operators $\cosh kz, \ldots$ are defined by their power-series expansions in k^2, and $\boldsymbol{z}_1$ is the unit vector in the z-direction. The corresponding results for the complex amplitudes of $\boldsymbol{u}$ and w are

$$\boldsymbol{U} = \cosh kz\nabla\Phi_0 + k^{-1}\sinh kz\nabla\Phi_1 - \kappa\sinh\kappa z\boldsymbol{\Psi}_0 - \cosh\kappa z\boldsymbol{\Psi}_1 \tag{2.10a}$$

and $$W = k\sinh kz\Phi_0 + \cosh kz\Phi_1 + \cosh\kappa z\nabla\cdot\boldsymbol{\Psi}_0 + \kappa^{-1}\sinh\kappa z\nabla\cdot\boldsymbol{\Psi}_1. \tag{2.10b}$$

Substituting (2.8*a*) and (2.10) into (2.6) and (2.7) and letting $k\delta \to 0$, we obtain

$$\Phi_0 = (g/\mathrm{i}\omega)(1+\lambda^2 k^2)Z, \quad \Phi_1 = -\mathrm{i}\omega Z, \quad \boldsymbol{\Psi}_0 = 2\delta^2\nabla\Phi_1, \tag{2.11a–c}$$

$$\boldsymbol{\Psi}_1 = \operatorname{sech}\kappa h[\cosh kh\nabla\Phi_0 - (k^{-1}\sinh kh - 2\delta\sinh\kappa h)\nabla\Phi_1], \tag{2.11d}$$

and $$\nabla\cdot(H\nabla Z) + KZ \equiv \mathscr{L}Z = 0, \tag{2.12}$$

where
$$H = \left[\frac{\sinh kh}{k} - \frac{\tanh\kappa h\cosh kh}{\kappa}\right](1+\lambda^2k^2) + K\left[\frac{1-\cosh kh}{k^2} + \frac{\tanh\kappa h\sinh kh}{\kappa k}\right], \tag{2.13}$$

∇ operates on both h and Z, k operates only on Z, and $\kappa \approx 1/\delta$. We emphasize that the limit $k\delta \to 0$ is asymptotic and that error factors of $1 + O(k^2\delta^2)$ are implicit.

3. Shallow-water approximation

The approximation provided by (2.12) and (2.13) is valid for arbitrary Kh, but in most applications viscosity is significant only if $Kh \ll 1$, in which domain (2.13) may be reduced to

$$H \simeq [h - \delta\tanh(h/\delta)][1 - \lambda^2\nabla^2 + O(Kh)] + O(Kh^2). \tag{3.1}$$

If both viscosity and capillarity are neglected ($\delta = \lambda = 0$), (3.1) reduces to $H = h$, and (2.12) reduces to the conventional shallow-water equation (Lamb 1932, §193). If $Z = Z(x)$ and $h \sim \sigma x \downarrow 0$, $\mathscr{L}Z = 0$ then is a second-order, ordinary differential equation with a regular singularity of exponents 0, 0 at $x = 0$, and the inviscid boundary condition $hZ' = 0$ may be satisfied by a regular (at $x = 0$) solution; see e.g. §186, 2° in Lamb (1932). Similar results hold locally, or through separation of variables, for a non-straight shoreline; see e.g. §193 in Lamb (1932). If $\lambda = 0$ and $\delta_* \ll h$, (3.1) reduces to $H = h - \delta$, and (2.12) reduces to the conventional shallow-water equation (Lamb 1932, §193) with a boundary-layer correction ($h \to h - \delta$). This interpretation also holds if $0 < \lambda \ll h$, but the boundary-layer approximation manifestly fails in the limit $h \downarrow 0$.

If $h \downarrow 0$ with $\delta_*, \lambda > 0$, (3.1) reduces to

$$H \sim \tfrac{1}{3}\delta^{-2}h^3(1 - \lambda^2\nabla^2) \quad (h/\delta \to 0). \tag{3.2}$$

Suppose, for simplicity, that $Z = Z(x)$ and $h \approx \sigma x \downarrow 0$ (the following argument is

† A $\boldsymbol{z}_1$ component of $\boldsymbol{A}$ is admissible but redundant.

qualitatively valid for, and may be quantitatively extended to, longshore variations that are slow compared with the variation normal to the shoreline). $\mathscr{L}Z = 0$ then is a fourth-order, ordinary differential equation with a regular singularity (at $x = 0$) of exponents 2, 1, 0, 0, and the method of Frobenius (Ince 1944, §§16.1–16.3) yields four linearly independent solutions with the limiting forms

$$Z_2 = \xi^2 + (\tfrac{1}{18})\,\xi^3 + \dots, \quad Z_1 = \xi + \tfrac{1}{4}\xi^2[\log\xi + (\tfrac{13}{6})] + \dots, \quad Z_{01} = 1 - \xi(\log\xi + 2) + \dots, \tag{3.3a–c}$$

and

$$Z_{02} = \log\xi + 3 - \tfrac{1}{2}\xi(\log^2\xi + 4\log\xi + \tfrac{5}{2}) + \dots, \tag{3.3d}$$

where

$$\xi = \left(\frac{3\mathscr{K}\delta^2}{\sigma^4\lambda^2}\right)h = \mathrm{e}^{\frac{1}{2}\mathrm{i}\pi}\left(\frac{\rho\nu\omega}{\sigma^3 T}\right)x. \tag{3.4}$$

The requirements that the free-surface displacement and integral of the shear stress remain finite as $\xi \downarrow 0$ imply

$$|Z| < \infty, \quad |Z' - \xi Z''| < \infty \quad (\xi \downarrow 0), \tag{3.5a, b}$$

which rule out the solutions Z_{01} and Z_{02}, respectively; accordingly, the admissible solution of (2.12) and (3.1) for $h \downarrow 0$ has the form

$$Z(x) = AZ_1 + BZ_2, \tag{3.6}$$

where Z_1 and Z_2 are the solutions of exponents 1 and 2, which vanish at $\xi = 0$.

The prediction of a fixed contact line ($Z = 0$ at $h = 0$) is consistent with the laboratory observations of wave reflection from a sloping beach by Mahony & Pritchard (1980), who report that there was 'very little movement of the shoreline'. This suggests that the present model may be adequate for laboratory configurations of sufficiently small scale, but we emphasize that the physical conditions at the contact line may be much more complicated than those implied by the present hypotheses of no slip and uniform surface tension and that, for whatever reasons, shoreline motion (*runup*) of long gravity waves is seldom negligible on real beaches.†

The limit $h/\delta \downarrow 0$ with $\lambda = 0$, which might have been expected to yield a useful approximation (cf. the limit $h \downarrow 0$ with $\delta_* = \lambda = 0$), is considered in the Appendix and is found to imply an essential singularity at $h = 0$, which, in turn, implies total absorption of an incoming wave. It follows from these predictions that capillarity must be included in a physically acceptable description of the limit $h \downarrow 0$ in a viscous fluid.

4. Inextensible surface

We now suppose that the surface is inextensible (presumably in consequence of contamination). The condition of zero tangential stress, (2.2c) or (2.6c), then is replaced by the condition of zero tangential velocity, which leads to the replacement of (2.11) and (2.13) by

$$\Phi_0 = (g/\mathrm{i}\omega)(1 + \lambda^2 \ell^2) Z, \quad \Phi_1 = -\mathrm{i}\omega Z - \nabla\cdot\boldsymbol{\Psi}_0, \tag{4.1a, b}$$

$$\boldsymbol{\Psi}_0 = (\kappa \sinh\kappa h)^{-1}[(\cosh\kappa h - \cosh \ell h)\nabla\Phi_0 - \mathrm{i}\omega \ell^{-1}\sinh \ell h \nabla Z], \quad \boldsymbol{\Psi}_1 = \nabla\Phi_0, \tag{4.1c, d}$$

and

$$H \approx [h - 2\delta\tanh(h/2\delta)][1 - \lambda^2\nabla^2 + O(Kh)] + O(Kh^2), \tag{4.2}$$

† The joint hypotheses of no slip and a moving contact line imply an infinite shear force, which may be rendered finite either by fixing the contact line, as in the present model, or by relaxing the condition of no slip. See Dussan V. & Davis (1974) and Dussan V. (1979).

wherein error factors of $1+O(k^2\delta^2)$ are implicit. Comparing (2.13) and (4.2), we find that the change in the tangential boundary condition from zero stress to zero velocity implies the replacement of δ by 2δ. This might have been anticipated in the boundary-layer approximation, but is perhaps less obvious for $h = O(\delta)$.

The qualitative discussion of §3 remains valid for an inextensible surface.

5. Dissipation integral

The mean rate at which wave energy decays in consequence of viscous dissipation is given by (Lamb 1932, §329 (12), wherein it can be shown that the net contribution of the surface integrals is negligible compared with that of the volume integral if $K\delta_* \ll 1$)

$$D = \rho\nu \iiint \langle(\nabla \times \boldsymbol{q})^2\rangle \,\mathrm{d}V, \tag{5.1}$$

where $\langle\ \rangle$ signifies a temporal average over $2\pi/\omega$. Substituting $\boldsymbol{q}$ from (2.4), averaging, and invoking $\nu = \omega|\delta|^2$ and (2.5*b*), we obtain

$$D = \tfrac{1}{2}\rho\omega|\delta|^{-2} \iiint |\boldsymbol{A}|^2 \,\mathrm{d}V. \tag{5.2}$$

Substituting the shallow-water approximation

$$\boldsymbol{A} = \delta \operatorname{sech}(h/\delta) \sinh(z/\delta)\, \mathbf{z}_1 \times \nabla\Phi_0, \tag{5.3}$$

which follows from (2.8*b*), (2.11*c, d*), $Kh \ll 1$, and $\kappa \approx 1/\delta$, into (5.2) and integrating over $-h < z < 0$, we obtain

$$D = \tfrac{1}{2}\rho\omega\delta_* \operatorname{Re}\left\{(1-\mathrm{i}) \iint |\nabla\Phi_0|^2 \tanh(h/\delta)\,\mathrm{d}S\right\}. \tag{5.4}$$

If the lengthscale of Φ_0 is $1/k$ and $h \approx \sigma x$ for $\sigma x = O(\delta)$, $\tanh(h/\delta)$ may be approximated by 1 in (5.4) to obtain

$$D = \tfrac{1}{2}\rho\omega\delta_* \iint |\nabla\Phi_0|^2 \,\mathrm{d}S[1+O(k\delta/\sigma)], \tag{5.5}$$

which is equivalent to D calculated from the boundary-layer approximation. This result is unexpected (in view of the violation of the boundary-layer assumption $h \gg \delta_*$), and we emphasize that it depends implicitly on capillary effects in the neighbourhood of $h = 0$, the absence of which would imply total absorption (see Appendix).

Repeating the reduction of (5.2) with (2.11*c, d*) replaced by (4.1*c, d*), we obtain (5.4) and (5.5) with δ replaced by 2δ, as might have been conjectured from (4.2).

6. Stokes edge wave

The dominant edge wave for an inviscid liquid on a uniform, gentle slope, $h = \sigma x$, $\sigma \ll 1$, is given by Lamb (1932, §260)

$$\zeta = a\,\mathrm{e}^{-kx} \cos(ky - \omega t), \quad \omega^2 = \sigma g k \quad (\sigma \ll 1). \tag{6.1a, b}$$

We seek the corresponding solution of (2.12) and (3.1) in the form

$$Z = aF(\xi)\,\mathrm{e}^{-kx+\mathrm{i}ky}, \quad \xi \equiv \frac{h}{\delta} = \frac{\sigma x}{\delta}, \tag{6.2a, b}$$

on the assumptions that $|\alpha| \ll 1$, $\beta = O(\alpha)$ and $\gamma = O(1)$, where

$$\alpha \equiv \frac{k\delta}{\sigma}, \quad \beta \equiv \frac{\omega^2}{\sigma g k} - 1, \quad \gamma \equiv \frac{\lambda\sigma}{\delta}. \tag{6.3a–c}$$

We anticipate that β, which appears as an eigenvalue, is complex.

Substituting (6.2) into (2.12) and invoking (3.1), $K \equiv \omega^2/g$ and (6.3), we obtain

$$(p\mathscr{L}_\gamma F')' - 2\alpha p \mathscr{L}_\gamma F' - \alpha p' \mathscr{L}_\gamma F + \alpha(1+\beta)F = 0, \tag{6.4}$$

where

$$p \equiv \xi - \tanh\xi, \quad \mathscr{L}_\gamma F \equiv F + 2\alpha\gamma^2 F' - \gamma^2 F''. \tag{6.5a, b}$$

The requirements that Z (see §3) and $\int_0^h U\,dz$ vanish at $h = 0$ imply

$$F = 0, \quad p\mathscr{L}_\gamma(F' - \alpha F) = 0 \quad (\xi = 0). \tag{6.6a, b}$$

The requirement that Z (6.2a) decay as $kx \uparrow \infty$ implies

$$F = 1 + o(\alpha\, e^{\alpha\xi}) \quad (\alpha\xi \to \infty). \tag{6.7}$$

Integrating (6.4) as a first-order differential equation for $p\mathscr{L}_\gamma F'$, with the third and fourth terms therein being regarded as $O(\alpha)$ forcing terms, and invoking (6.6b), we obtain

$$p\mathscr{L}_\gamma F' = -\alpha \mathscr{F}(\xi)\, e^{2\alpha\xi}, \quad \mathscr{F} = \int_0^\xi [(1+\beta)F(\eta) - p'(\eta)\,\mathscr{L}_\gamma F(\eta)]\, e^{-2\alpha\eta}\, d\eta. \tag{6.8a, b}$$

Integrating (6.8) and invoking (6.7), we obtain

$$\mathscr{L}_\gamma F = 1 - \alpha \int_{\xi_1}^{\xi} \frac{\mathscr{F}(\eta)\, e^{2\alpha\eta}}{p(\eta)}\, d\eta, \tag{6.9}$$

where ξ_1 is a constant of integration (see below). Invoking (6.5b), integrating (6.9), *qua* second-order differential equation, and invoking (6.6a), we obtain

$$F(\xi) = F_0(\xi) - \frac{\alpha}{2\gamma^2\mu} \int_0^\infty [e^{-\mu|\xi-\eta|} - e^{-\mu(\xi+\eta)}]\, e^{\alpha(\xi-\eta)}\, d\eta \int_{\xi_1}^{\eta} \frac{\mathscr{F}(\zeta)\, e^{2\alpha\zeta}}{p(\zeta)}\, d\zeta, \tag{6.10}$$

where

$$F_0 = 1 - e^{-(\mu-\alpha)\xi}, \quad \mu = (\alpha^2 + \gamma^{-2})^{\frac{1}{2}}. \tag{6.11a, b}$$

The integral equation (6.10) may be solved by iteration, starting from the first approximation $F = F_0$ and adjusting β at each stage of the iteration to satisfy (6.7). Substituting (6.11a) into (6.8b) and invoking (6.7), we obtain the first approximations

$$\beta = 2\alpha\left(\frac{\mu+\alpha}{\mu-\alpha}\right)\left\{-1 + \frac{1}{\mu+\alpha} + \alpha[\psi(1+\tfrac{1}{2}\alpha) - \psi(\tfrac{1}{2}+\tfrac{1}{2}\alpha)]\right\}[1+O(\alpha)] \tag{6.12a}$$

$$= 2\alpha[-1 + \gamma + O(\alpha)], \tag{6.12b}$$

where ψ is the logarithmic derivative of the gamma function, and

$$\mathscr{F} = \left(\frac{1+\beta}{\mu+\alpha}\right) e^{-(\mu+\alpha)\xi} - \frac{\beta}{2\alpha} e^{-2\alpha\xi} - \int_\xi^\infty e^{-2\alpha\eta}\,\mathrm{sech}^2\,\eta\, d\eta \tag{6.13a}$$

$$= e^{-2\alpha\xi}(\tanh\xi - \gamma) + \gamma\, e^{-\xi/\gamma} + O(\alpha). \tag{6.13b}$$

Substituting (6.12b) into (6.3b) and invoking (6.3a, c), we obtain the complex frequency

$$\omega = (\sigma g k)^{\frac{1}{2}}[1 + k\lambda - \sigma^{-1}k\delta + O(\alpha^2)]. \tag{6.14}$$

The damping ratio, $k\delta_*/\sigma$, as already shown in §5, is equal to that obtained through a boundary-layer calculation (cf. Guza & Davis 1974).

We proceed on the hypothesis that $\mathscr{L}_\gamma F \sim F$ as $\xi/\gamma \to \infty$. It then follows from (6.9) that

$$F \sim 1-\alpha \int_{\xi_1}^{\xi} \frac{\mathscr{F}(\eta)\,\mathrm{e}^{2\alpha\eta}}{p(\eta)}\,\mathrm{d}\eta \quad (\xi/\gamma \to \infty). \tag{6.15}$$

Substituting (6.5*a*) and (6.13*b*) into (6.15) and anticipating $|\xi_1/\gamma| \gg 1$ (see below), we obtain

$$F \sim 1-\alpha \int_{\xi_1}^{\xi} \left(\frac{\tanh\eta-\gamma}{\eta-\tanh\eta}\right)\mathrm{d}\eta \sim 1-\alpha(1-\gamma)\log\left(\frac{\xi-1}{\xi_1-1}\right) \quad (|\xi,\xi_1| \gg 1), \tag{6.16}$$

from which $\mathscr{L}_\gamma F \sim F$ may be confirmed.

Returning to (6.4) and letting $\mathscr{L}_\gamma F \sim F$ and $p \sim \xi-1$ therein, we obtain

$$(\xi-1)F''+[1-2\alpha(\xi-1)]F'+\alpha\beta F = 0. \tag{6.17}$$

The solution of this confluent hypergeometric equation (Abramowitz & Stegun 1964, §13.1.6), subject to (6.7), yields the outer approximation

$$F = \Gamma(1-\tfrac{1}{2}\beta)\,U[-\tfrac{1}{2}\beta, 1, 2\alpha(\xi-1)] \tag{6.18a}$$

$$= 1+\tfrac{1}{2}\beta\{\log[2C\alpha(\xi-1)]+\psi(1-\tfrac{1}{2}\beta)-\psi(1)\}+O[\alpha\beta^2(\xi-1)], \tag{6.18b}$$

where $C = 1.781\ldots$ is Euler's constant. Matching (6.18*b*) to (6.17), we confirm (6.12*b*) and obtain

$$\xi_1 = \frac{1}{2C\alpha}+\frac{\pi^2(1-\gamma)}{12C}+1+O(\alpha). \tag{6.19}$$

Finally, we let $\alpha(\xi-1) \to \infty$ in (6.18*a*) and invoke (6.2*a*) to obtain

$$Z \sim a\Gamma(1-\tfrac{1}{2}\beta)\,[2\sigma^{-1}k(h-\delta)]^{k\lambda-\alpha}\,\mathrm{e}^{-kx+iky} \quad (kh/\sigma \uparrow \infty), \tag{6.20}$$

which describes the modification of the edge wave decay by small capillarity and viscosity. This last result is qualified by the shallow-water approximation; however, just as (6.1) can be shown to provide a valid inviscid approximation in $kh \ll 1$, within an error factor of $1+O(\sigma/kh_1)$, for a smooth profile of asymptotic depth h_1 (Miles 1989), so also can (6.20).† An asymptotic approximation for $kh = O(1)$ that matches (6.21) in $kh \ll 1$ could be constructed along the lines of §4 in this last reference, but Z is exponentially small in $kh = O(1)$, and (6.20) should suffice for most cases of oceanographic interest.

This paper is dedicated to GKB as a token of affection and esteem.

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Appendix. The limit $h/\delta \downarrow 0$, capillarity absent

The limit $h/\delta \downarrow 0$ with $\lambda = 0$ reduces (3.1) to

$$H \sim \tfrac{1}{3}\delta^{-2}h^3 \quad (h \sim \sigma x \downarrow 0, \quad \lambda = 0). \tag{A 1}$$

† If $h(x)$ is not smooth, as in a typical wave tank, in which h' decreases discontinuously from σ to 0 at the toe of the beach, the dominant edge wave must be accompanied by higher modes in order to render both Z and Z' continuous at the toe.

Then, if $Z = Z(x)$, $x = 0$ is an irregular singular point of $\mathscr{L}Z = 0$, and there exists a pair of linearly independent solutions, $Z_{\pm}$, that describe waves moving in the $\pm x$-directions and exhibit the limiting behaviours

$$Z_{\pm} = O\left\{H^{-\frac{1}{4}}\exp\left[\pm \mathrm{i}\int (K/H)^{\frac{1}{2}}\mathrm{d}x\right]\right\} = \mathrm{O}\{\mathrm{x}^{-\frac{3}{4}}\exp\left[\pm \delta_*(3K/\sigma^3)^{\frac{1}{2}}x^{-\frac{1}{2}}\right]\} \quad (x \downarrow 0), \tag{A 2}$$

wherein the alternative signs are vertically ordered. The outgoing wave Z_+ is exponentially infinite, and therefore physically inadmissible, while the incoming wave Z_- is totally absorbed and vanishes at $x = 0$. It follows that the approximation $\lambda = 0$ with $\delta_* > 0$ is not uniformly valid as $h \downarrow 0$, and capillarity must be included to obtain physically meaningful results.

(The essential role of capillarity for $h \ll \delta_*$ is to raise the order of $\mathscr{L}Z$. This is associated with the fact that capillarity dominates gravity in governing wave propagation for sufficiently small wavenumbers. The elimination of a physically unacceptable singularity through an increase of order of the differential equation is reminiscent of, but more complicated than, the elimination of the critical-layer singularity in the Orr–Sommerfeld equation through the introduction of viscosity.)

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